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KINETIC MODEL OF BUBBLY FLOW

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A kinetic approach based on the approximate calculation of the fluid flow potential and formulation of Hamilton's equations for generalized coordinates and momenta of bubbles is employed to describe processes of collective interaction of gas bubbles moving in an inviscid incompressible fluid. Kinetic equations governing the evolution of the distribution function of bubbles are derived. These equations are similar to Vlasov equations.

Kinetic approaches for the description of fluid flows with gas bubbles have been developed in a number of recent papers [1–4]. Some of the systems of equations obtained are similar in structure to Vlasov equations which are used to describe plasma flows. In the derivation of these equations, Hamilton's ordinary differential equations that describe the motion of individual particles are employed. If gas bubbles moving in an inviscid incompressible fluid are treated as particles, then for the derivation of the above-mentioned ordinary differential equations, one needs to know the fluid flow potential in the region between the particles. For the simplified situation where the bubbles are considered incompressible, an approximate Hamiltonian describing the motion of the bubbles for a rarefied bubbly medium was obtained by Russo and Smereka [4] and used to derive a system of kinetic equations governing the evolution of the one-particle distribution function. The assumption on the incompressibility of bubbles can be used to describe real flows with bubbles of sufficiently small size where the surface tension, which maintains the shape of the bubbles, is considerably greater than the variations of the hydrodynamic pressure. This model can be used for description of concentration waves for small pressure differences.

The motion of a system of compressible bubbles in a fluid is often modeled using averaged equations, supplemented with the Rayleigh–Lamb equation for a single bubble. Various models of this type differing in additional terms of equations describing real effects are discussed in [5, 6]. Certain hydrodynamic effects related to motion of bubbles in a fluid were considered in a monograph by Lavrent'ev and Shabat [7]. A kinetic approach for modeling bubbly flows in which the evolution of the bubble distribution function is governed by equations similar to Boltzmann equations or Vlasov equations, allows one to describe the motion more thoroughly and to derive average equations using a regular procedure. In particular, this approach can, in principle, yield certain basic relations that are postulated in a hydrodynamic description.

In the present paper, we derive a system of kinetic equations for bubbly flow that describes the motion of compressible gas bubbles in an inviscid incompressible fluid. We first formulate the system of Hamilton's equations for generalized coordinates of spherical bubbles (spatial coordinates of the centers and radii) and the corresponding momenta. This system is easily written if the potential of the irrotational fluid flow in the region between the bubbles is known. For approximate calculation of the potential, we use an asymptotic expansion of the solution of the Laplace equation in the small parameter — the ratio of the mean radius of the bubbles to the mean distance between them. Lagrange equations describing the evolution of the bubble system follow from the law of conservation of energy. By a standard transformation, these equations are transformed to Hamilton's equations. We then write an equation for the N-particle distribution function,

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average this equation over the coordinates and momenta of $N-1$ particles, and thus obtain Vlasov equations for the one-particle distribution function. In the present paper, we obtain kinetic equations that correspond to the leading term in the asymptotic expansion of the fluid flow potential. Steady solutions of these equations are considered.

1. Hamilton's Equations for Gas Bubbles Moving in a Fluid. We consider N spherical gas bubbles moving in an unbounded inviscid incompressible fluid. We assume that the fluid flow in the region between bubbles is irrotational and the velocity vector vanishes as $|\boldsymbol{x}| \to \infty$. The fluid velocity potential $\varphi(t, \boldsymbol{x})$ is a solution of the following boundary-value problem:

$$\Delta\varphi = 0, \quad \boldsymbol{x} \in \Omega = \mathbb{R}^3 \setminus \bigcup_j B_j; \quad \frac{\partial\varphi}{\partial n}\Big|_{\Gamma_j} = \boldsymbol{v} \cdot \boldsymbol{n}_j + s_j; \quad \nabla\varphi \to 0, \quad |\boldsymbol{x}| \to \infty. \tag{1.1}$$

Here $\boldsymbol{v}_j(t) = \boldsymbol{x}_j'(t)$ are the velocities of the centers of spherical bubbles, $s_j = b_j'(t)$ are the velocities of expansion of the bubbles, $\boldsymbol{x}_j(t)$ are the radius-vectors of the centers, $b_j(t)$ are the radii of the bubbles $(j = 1, \ldots, N)$, B_j and Γ_j are a ball and a sphere of radius $b_j(t)$ with center at the point $\boldsymbol{x} = \boldsymbol{x}_j(t)$, respectively, and $\boldsymbol{n}_j$ is the normal to Γ_j directed to the fluid; prime denotes derivative with respect to time.

The unknown potential of the irrotational flow can be written as

$$\varphi = \sum_{j=1}^{N} (\boldsymbol{v}_j \boldsymbol{\psi}_j + s_j \varphi_j), \tag{1.2}$$

where, in view of (1.1), the harmonic functions $\boldsymbol{\psi}_j(t, x)$ and $\varphi_j(t, x)$ satisfy the conditions

$$\frac{\partial\boldsymbol{\psi}_j}{\partial n}\Big|_{\Gamma_k} = \delta_{jk}\boldsymbol{n}_j, \qquad \frac{\partial\varphi_j}{\partial n}\Big|_{\Gamma_k} = \delta_{jk} \tag{1.3}$$

and their gradients vanish at infinity ($\delta_{jk} = 0$ for $j \neq k$ and $\delta_{jj} = 1$).

Let $\bar{b}$ be the mean radius of a bubble and $\bar{R}$ the mean distance between bubbles. In what follows, we consider a rarefied bubbly fluid for which $\beta = \bar{b}(\bar{R})^{-1} \ll 1$. Using asymptotic expansions in a small parameter, we obtain approximate values of the flow potential near each bubble.

The function $\varphi_{j0} = -b_j^2/r_j$ ($r_j = |\boldsymbol{x} - \boldsymbol{x}_j|$), which is harmonic outside the ball B_j, satisfies the condition $\partial\varphi_j/\partial n = 1$ on Γ_j and the decay condition for large $|\boldsymbol{x}|$. In the neighborhood of the ith bubble, we have $r_j^2 = r_{ij}^2|\boldsymbol{n}_{ji} - r_{ij}^{-1}(\boldsymbol{x} - \boldsymbol{x}_i)|^2 = r_{ij}^2(1 - 2\cos\theta_{ji}(r_i/r_{ij}) + (r_i/r_{ij})^2)$, $r_{ij} = |\boldsymbol{x}_i - \boldsymbol{x}_j|$, $\boldsymbol{n}_{ji} = (\boldsymbol{x}_j - \boldsymbol{x}_i)r_{ij}^{-1}$, $\cos\theta_{ji} = (\boldsymbol{x} - \boldsymbol{x}_i)\boldsymbol{n}_{ji}r_i^{-1}$, and $r_i = |\boldsymbol{x} - \boldsymbol{x}_i|$.

Using the generating function for Legendre polynomials [8] and the representation for r_j written above, we obtain the following series expansion of the function φ_{j0}

$$\varphi_{j0} = -b_j \frac{b_j}{r_{ij}} \sum_{n=0}^{\infty} \left(\frac{r_i}{b_i}\right)^n \left(\frac{b_i}{r_{ij}}\right)^n P_n(\cos\theta_{ji}), \tag{1.4}$$

which is valid in the neighborhood of the ith bubble ($r_i/r_{ij} < 1$). Equation (1.4) for $r_i = b_i$ gives the expansion of the trace of the function φ_{j0} on Γ_i in powers of b_i/r_{ij}. We note that the quantity b_i/r_{ij} is of order $\beta \ll 1$. In (1.4), $P_n(t)$ is a Legendre polynomial of the nth degree. It is known that, along with $r^n P_n(\cos\theta)$, the function $r^{-(n+1)}P_n(\cos\theta)$ also satisfies the Laplace equation. We now introduce the harmonic function

$$\varphi_{ji} = -\frac{b_j^2}{r_{ij}} \sum_{n=1}^{\infty} \frac{n}{n+1} \left(\frac{b_i}{r_i}\right)^{n+1} \left(\frac{b_i}{r_{ij}}\right)^n P_n(\cos\theta_{ji}). \tag{1.5}$$

It is easy to verify that

$$\frac{\partial}{\partial r_i}(\varphi_{j0} + \varphi_{ji}) = 0 \tag{1.6}$$

for $r_i = b_i$. We consider the function

$$\varphi_j = \varphi_{j0} + \sum_{i \neq j} \varphi_{ji}^{(2)}, \tag{1.7}$$

where $\varphi_{ji}^{(2)}$ is defined by formula (1.5) where the total sum of the series is replaced by the partial sum of terms up to $n = 2$. It is easy to see that $\partial\varphi_j/\partial n\Big|_{\Gamma_i} = \delta_{ik} + O(\bar{b}/\bar{R})^4$ by virtue of Eqs. (1.6), (1.7) and the estimates $\varphi_{ji}\Big|_{\Gamma_k} = O(\bar{b}/\bar{R})^4$ and $\nabla\varphi_{ji}\Big|_{\Gamma_k} = O(\bar{b}/\bar{R})^4$. Here $i \neq k$. Hence, the function φ_j defined by formula (1.7) is an approximate solution of problem (1.3) for Laplace equation. The difference of the approximate and the exact solutions is of order not lower than $(\bar{b}/\bar{R})^4$.

We consider the harmonic function $\boldsymbol{\psi}_{j0} = (b_j^3/2)\nabla(r_j^{-1}) = -(b_j/r_j)^3(\boldsymbol{x} - \boldsymbol{x}_j)/2$ which satisfies the boundary condition $(\partial\boldsymbol{\varphi}_{j0}/\partial n)\Big|_{\Gamma_j} = \boldsymbol{n}_j$ on the surface of the jth bubble. In the neighborhood of the ith bubble, $\boldsymbol{\psi}_{j0}$ admits the following approximate representation by a partial Taylor series with remainder of order $(\bar{b}/\bar{R})^4$:

$$\boldsymbol{\psi}_{j0} = (b_j/2)[(b_j/r_{ij})^2\boldsymbol{n}_{ji} - (b_j/r_{ij})^2(r_i/b_i)(b_i/r_{ij})B_{ij}\langle(\boldsymbol{x} - \boldsymbol{x}_i)r_i^{-1}\rangle] + O((\bar{b}/\bar{R})^4),$$

where $B_{ij} = I - 3\boldsymbol{n}_{ji} \otimes \boldsymbol{n}_{ji}$, I is the unit matrix, and $\boldsymbol{a} \otimes \boldsymbol{b}$ is the diadic of two vectors. We introduce the harmonic function

$$\boldsymbol{\psi}_{ji} = (b_j/4)(b_j/r_{ij})^2(b_i/r_{ij})(b_i/r_i)^2B_{ij}\langle(\boldsymbol{x} - \boldsymbol{x}_i)r_i^{-1}\rangle. \tag{1.8}$$

It is easy to verify that $\partial(\boldsymbol{\psi}_{j0} + \boldsymbol{\psi}_{ji})/\partial n\Big|_{\Gamma_i} = 0$ and to show, using this equality, that the function

$$\boldsymbol{\psi}_j = \boldsymbol{\psi}_{j0} + \sum_{i\neq j}\boldsymbol{\psi}_{ji} \tag{1.9}$$

is an approximate solution of problem (1.3) and its difference from the exact solution is of order not lower than $(\bar{b}/\bar{R})^4$.

As a result, we have obtained an approximate representation of the fluid velocity potential for specified velocities of motion and expansion of bubbles. This enables us to calculate the kinetic energy of the fluid:

$$T = \frac{\rho}{2}\iiint_\Omega |\boldsymbol{u}|^2\,d\Omega = -\frac{\rho}{2}\sum_{i=1}^{N}\iint_{\Gamma_i}\varphi\frac{\partial\varphi}{\partial n}\,d\Gamma = -\frac{\rho}{2}\sum_{i=1}^{N}\iint_{\Gamma_i}\varphi(\boldsymbol{v}_i\boldsymbol{n}_i + s_i)\,d\Gamma$$

(the normal is directed to the fluid and ρ is the density of the fluid).

Using (1.2), we can write T as

$$T = -\frac{\rho}{2}\sum_{j=1}^{N}\sum_{i=1}^{N}[(\boldsymbol{v}_i, A_{ji}\boldsymbol{v}_i) + s_j\boldsymbol{d}_{ij}\boldsymbol{v}_i + \boldsymbol{v}_j\boldsymbol{c}_{ij}s_i + s_je_{ij}s_i], \quad A_{ji} = \iint_{\Gamma_i}\boldsymbol{\psi}_j \otimes \boldsymbol{n}_i\,d\Gamma,$$

$$\boldsymbol{d}_{ij} = \iint_{\Gamma_i}\varphi_j\boldsymbol{n}_i\,d\Gamma, \quad \boldsymbol{c}_{ij} = \iint_{\Gamma_i}\boldsymbol{\psi}_j\,d\Gamma, \quad e_{ij} = \iint_{\Gamma_i}\varphi_j\,d\Gamma.$$

To calculate the coefficients of this quadratic form, it suffices to know the values of the potentials $\boldsymbol{\psi}_j$ and φ_j on Γ_i.

Using Eqs. (1.5) and (1.7)–(1.9), we calculate the coefficients of the quadratic form T (when calculating the integrals over the kth sphere in the sums in Eqs. (1.7) and (1.9), we need to consider only the terms $\boldsymbol{\psi}_{jk}$ and φ_{jk} since the contribution of the remaining terms is of order higher than β^3). As a result, we obtain the following expression for the kinetic energy of the fluid:

$$T = \frac{\rho}{2}\Big[\sum_{i=1}^{N}b_i^3\Big(\frac{2}{3}\pi|\boldsymbol{v}_i|^2 + 4\pi s_i^2\Big) + \sum_{i=1}^{N}\sum_{j\neq i}\Big(4\pi b_i^2b_j\frac{b_j}{r_{ij}}s_is_j + 2\pi b_i^3\Big(\frac{b_j}{r_{ij}}\Big)^2s_j(\boldsymbol{n}_{ij}\boldsymbol{v}_i)$$
$$+ 2\pi b_j^3\Big(\frac{b_i}{r_{ij}}\Big)^2s_i(\boldsymbol{n}_{ji}\boldsymbol{v}_j) + \pi b_i^3\Big(\frac{b_j}{r_{ij}}\Big)^3(\boldsymbol{v}_j, B_{ij}\boldsymbol{v}_i)\Big)\Big].$$

In this formula, the first sum comprises terms of zeroth order in β and the second sum includes terms of orders β, β^2, and β^3. A formula for the kinetic energy that takes account of terms of order not higher than β^2 was given in [9, 10].

In the calculation of the total kinetic energy of the system "fluid–gas bubbles," we do not take account of the kinetic energy of the gas, because the mass of the gas is small compared to the mass of the fluid. The law of conservation of energy for the fluid that occupies the region between the bubbles is written as

$$\frac{dT}{dt} = \sum_{i=1}^{N} \iint_{\Gamma_i} (p - P) u_n \, d\Gamma.$$

Here p is the pressure in the fluid, $P = \text{const}$ is the pressure at infinity, and $u_n = \boldsymbol{u}\boldsymbol{n} = \partial\varphi/\partial n$ ($\boldsymbol{n}$ is the normal directed to the fluid).

We note that the assumption of sphericity of the bubbles used for an approximate description of the flow simplifies the problem substantially and makes it possible to determine the main contribution to the variation of the fluid flow potential caused by oscillations of the bubble volume. In the exact formulation of the problem, the bubble shape must be obtained as a result of solving the problem with unknown fluid–gas boundary from the condition of equality of the pressures in the gas and in the fluid on Γ_i. Therefore, in an approximate description, the formulation of the problem is modified: on the boundary we require equality of the pressures averaged over the surface. Let τ be the volume of a bubble. The state of the gas inside a bubble will be described approximately under the assumption that the density ρ and the pressure p are constant over the volume. Let the state of the gas be described by the equation of state $p = f_1(\rho)$ (isentropic process). The conservation of mass for the gas in a bubble yields the relation $\rho\tau = \rho_0\tau_0$. Here ρ_0 and p_0 are the density and pressure of the gas at the initial time and τ_0 is the initial volume. We assume that, initially, all bubbles have the same mass m_0. Then, they have equal masses at any time and, as was mentioned above, the pressure in a bubble is determined by its volume: $p = f(\tau) = f_1(m_0/\tau)$. Using (1.2) and (1.3), we obtain

$$\sum_{i=1}^{N} \iint_{\Gamma_i} p \frac{\partial\varphi}{\partial n} \, d\Gamma = \sum_{i=1}^{N} \left(\boldsymbol{v}_i \iint_{\Gamma_i} p\boldsymbol{n} \, d\Gamma + s_i \iint_{\Gamma_i} p \, d\Gamma \right). \tag{1.10}$$

But the integral in the first term of the right side of this equality is zero (the total force exerted on a bubble is equal to zero by virtue of the law of conservation of momentum).

Taking into account that the pressure depends only on the bubble volume, we can write Eq. (1.10) in the form

$$\sum_{i=1}^{N} \iint_{\Gamma_i} (p - P) \frac{\partial\varphi}{\partial n} \, d\Gamma = \sum_{i=1}^{N} (p(\tau_i) - P) \frac{d\tau_i}{dt} = -\sum_{i=1}^{N} \frac{d\varepsilon(\tau_i)}{dt}, \quad \varepsilon(\tau) = \int_{\tau}^{\infty} p(\tau) \, d\tau + P\tau.$$

It is well known that the problem of motion of a fluid with bubbles is a Lagrangian problem in the case where the flow of the fluid is completely determined by bubble motion [9]. Exactly this situation is considered in the present paper. Using the law of conservation of energy, we obtain the Lagrange equations for the generalized coordinates and the corresponding velocities:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right) - \frac{\partial L}{\partial \boldsymbol{q}_i} = 0, \qquad L = T - U, \qquad U = \sum_{i=1}^{N} \varepsilon(\tau_i). \tag{1.11}$$

Here $\boldsymbol{q}_i = (\boldsymbol{x}_i, b_i)$ ($i = 1, \ldots, N$) are the vectors with four components. Instead of the generalized coordinate b_i (bubble radius), it is convenient to introduce the coordinate $\sigma_i = b_i^2$, which is proportional to bubble's surface area, and the velocity $\Sigma_i = \sigma_i' = 2b_i s_i$. In this case, the Lagrange equation retains the form (1.11) with $\boldsymbol{q}_i = (\boldsymbol{x}_i, \sigma_i)$.

In the present paper, we shall consider only the leading term in the expansion of the kinetic energy in the parameter β. We write the Lagrangian keeping only terms of zeroth and first order in the small parameter:

$$L = \frac{\rho}{2}\left(\sum_{i=1}^{N} \left(\frac{2}{3}\pi\sigma_i^{3/2}|\boldsymbol{v}_i|^2 + \pi\sigma_i^{1/2}\Sigma_i^2 \right) + \sum_{i=1}^{N}\sum_{j\neq i} \pi\sigma_i^{1/2}\sigma_j^{1/2}\frac{\Sigma_i\Sigma_j}{r_{ij}} \right) - \sum_{i=1}^{N} \varepsilon\left(\frac{4}{3}\pi\sigma_i^{3/2}\right).$$

To convert to Hamilton's equations, we introduce the Hamiltonian

$$H = T + U \tag{1.12}$$

and the generalized momenta $\boldsymbol{p}_i$ and Q_i:

$$\boldsymbol{p}_i = \frac{\partial L}{\partial \boldsymbol{v}_i} = \frac{2}{3}\pi\rho\sigma_i^{3/2}\boldsymbol{v}_i, \quad Q_i = \frac{\partial L}{\partial \Sigma_i} = \pi\rho\sigma_i^{1/2}\Sigma_i + \pi\rho\sigma_i^{1/2}\sum_{i\neq j}\sigma_j^{1/2}\frac{\Sigma_j}{r_{ij}}.$$

Then, the generalized velocities $\boldsymbol{v}_i$ and Σ_i are expressed in terms of the generalized momenta $\boldsymbol{p}_i$ and Q_i and are substituted into (1.12). Since $\Sigma_i = Q_i/(\pi\rho\sigma_i^{1/2}) + O(\beta)$, up to terms of order β^2, we have

$$\boldsymbol{v}_i = \boldsymbol{p}_i\Big(\frac{2}{3}\pi\rho\sigma_i^{3/2}\Big)^{-1} = \frac{2\boldsymbol{p}_i}{\tau_i\rho}, \quad \Sigma_i = (\pi\rho\sigma_i^{1/2})^{-1}\Big(Q_i - \sigma_i^{1/2}\sum_{i\neq j}\frac{Q_j}{r_{ij}}\Big). \tag{1.13}$$

Using these formulas, we obtain the approximate expression for the Hamiltonian:

$$H = \sum_{i=1}^{N}\Big(\frac{|\boldsymbol{p}_i|^2}{\rho\tau_i} + \frac{Q_i^2}{2\pi\rho\sigma_i^{1/2}}\Big) - \sum_{i=1}^{N}\sum_{j\neq i}\frac{Q_iQ_j}{2\pi\rho r_{ij}} + \sum_{i=1}^{N}\varepsilon(\tau_i), \qquad \tau_i = \frac{4\pi\sigma_i^{3/2}}{3}. \tag{1.14}$$

In Hamiltonian form, the equations of motion for the bubbles are

$$\frac{d\boldsymbol{x}_i}{dt} = H_{\boldsymbol{p}_i}, \quad \frac{d\boldsymbol{p}_i}{dt} = -H_{\boldsymbol{x}_i}, \quad \frac{d\sigma_i}{dt} = H_{Q_i}, \quad \frac{dQ_i}{dt} = -H_{\sigma_i}. \tag{1.15}$$

For $N = 1$ (motion of a single bubble), $H = |\boldsymbol{p}|^2/(\rho\tau) + Q^2/(2\pi\rho\sigma^{1/2}) + \varepsilon(\tau)$ and Hamilton's equations have the form

$$\frac{d\boldsymbol{x}}{dt} = \frac{2\boldsymbol{p}}{\rho\tau}, \quad \frac{d\boldsymbol{p}}{dt} = 0, \quad \frac{d\sigma}{dt} = \frac{Q}{\pi\rho\sigma^{1/2}}, \quad \frac{dQ}{dt} = \frac{Q^2}{4\pi\rho\sigma^{3/2}} + 2\pi\sigma^{1/2}\Big(p(\tau) - P + \frac{|\boldsymbol{p}|^2}{\rho\tau^2}\Big). \tag{1.16}$$

From the vector equations it follows that the velocity of a bubble is given by the relations $\boldsymbol{v} = 2\boldsymbol{p}_0/(\rho\tau)$, where $\boldsymbol{p}_0 = \rho\tau_0\boldsymbol{v}_0/2$ (τ_0 and $\boldsymbol{v}_0$ are the initial values of volume and velocity). From the third equation of system (1.16) we express Q in terms of the bubble radius b and s: $Q = 2\pi\rho b^2 s$, where $s = b'$. The last equation of (1.16) reduces to the Rayleigh–Lamb equation

$$b\frac{ds}{dt} + \frac{3s^2}{2} = \frac{p(\tau) - P}{\rho} + \frac{|\boldsymbol{v}_0|^2\tau_0^2}{4\tau^2} \tag{1.17}$$

for a moving bubble. We note that the order of Eq. (1.17) can be reduced because the Hamiltonian system always has the integral $H = H_0 = \text{const}$. After certain transformations we obtain the equations

$$\frac{d\boldsymbol{x}}{dt} = \frac{\tau_0\boldsymbol{v}_0}{\tau}, \qquad \frac{db}{dt} = \pm\Big(\frac{2(H_0 - \varepsilon(\tau))}{3\tau\rho} - \frac{|\boldsymbol{v}_0|^2\tau_0^2}{6\tau^2}\Big)^{1/2}, \tag{1.18}$$

which can be integrated in quadratures. The solution of Eqs. (1.18) describes periodic oscillations of a bubble in rectilinear motion with variable velocity. System (1.18) differs from a similar system in [11] in an additional term due to compressibility of the bubble.

Generally, system (1.15) admits the following integrals (the laws of conservation of momentum, angular momentum, and energy):

$$\sum_{i=1}^{N}\boldsymbol{p}_i = \boldsymbol{p}_0 = \text{const}, \quad \sum_{i=1}^{N}\boldsymbol{x}_i \times \boldsymbol{p}_i = \boldsymbol{q}_0 = \text{const}, \quad H = H_0 = \text{const}.$$

The law of conservation of momentum follows from the invariance of the Hamiltonian with respect to the translation transformation $\boldsymbol{x}_j' = \boldsymbol{x}_j + \boldsymbol{a}$, where $\boldsymbol{a}$ is an arbitrary constant vector. The law of conservation of angular momentum is a consequence of the invariance of the Hamiltonian with respect to the simultaneous rotation of $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$ ($i = 1, \ldots, N$).

2. Derivation of Kinetic Equations. In the present paper, we consider the collisionless case assuming that the bubbly medium is sufficiently rarefied, so that over the characteristic time the bubbles collide with each other very rarely. The basic equation of the kinetic model is the law of conservation of the number of particles during their motion. We now consider the N-particle distribution function $f^{(N)}(t, \boldsymbol{x}_1, \sigma_1, \boldsymbol{p}_1, Q_1, \ldots, \boldsymbol{x}_N, \sigma_N, \boldsymbol{p}_N, Q_N)$, which, in view of (1.15), satisfies the Liouville equation [12]

$$f_t^{(N)} + \sum_{k=1}^{N} \Big(\operatorname{div}_{\boldsymbol{x}_k}(H_{\boldsymbol{p}_k} f^{(N)}) - \operatorname{div}_{\boldsymbol{p}_k}(H_{\boldsymbol{x}_k} f^{(N)}) + (H_{Q_k} f^{(N)})_{\sigma_k} - (H_{\sigma_k} f^{(N)})_{Q_k}\Big) = 0 \qquad (2.1)$$

and vanishes for large $\boldsymbol{x}_k$, $\boldsymbol{p}_k$, σ_k, and Q_k. Equation (2.1) is an analogue of the hydrodynamic continuity equation for motion along the trajectories of system (1.15). The unknown function in this equation depends on a large number of independent variables, and construction of its solution is, in fact, equivalent to integration of Eqs. (1.15). Therefore, using the averaging method, we deduce simpler equations that describe the evolution of the one-particle distribution function defined by

$$f^{(1)}(t, \boldsymbol{x}_1, \sigma_1, \boldsymbol{p}_1, Q_1) = \int f^{(N)}\, d\Omega_1,$$
$$d\Omega_1 = d\boldsymbol{x}_2 \cdots d\boldsymbol{x}_N\, d\boldsymbol{p}_2 \cdots d\boldsymbol{p}_N d\sigma_2 \cdots d\sigma_N\, dQ_2 \cdots dQ_N. \qquad (2.2)$$

Multiplying Eq. (2.1) by $d\Omega_1$ and integrating over the generalized coordinates and the momenta of $N-1$ particles, we obtain the following equation for the function $f^{(1)}$:

$$f_t^{(1)} + \operatorname{div}_{\boldsymbol{x}_1}\Big(\int H_{\boldsymbol{p}_1} f^{(N)}\, d\Omega_1\Big) - \operatorname{div}_{\boldsymbol{p}_1}\Big(\int H_{\boldsymbol{x}_1} f^{(N)}\, d\Omega_1\Big)$$
$$+ \Big(\int H_{Q_1} f^{(N)}\, d\Omega_1\Big)_{\sigma_1} - \Big(\int H_{\sigma_1} f^{(N)}\, d\Omega_1\Big)_{Q_1} = 0. \qquad (2.3)$$

Let us calculate the integral terms of this equation. Since $H_{\boldsymbol{p}_1} = \boldsymbol{v}_1$, from (1.13) and (2.1) we obtain

$$\int H_{\boldsymbol{p}_1} f^{(N)}\, d\Omega_1 = \frac{2\boldsymbol{p}_1 f^{(1)}}{\tau_1 \rho}.$$

Similarly, using the equality $H_{Q_1} = \Sigma_1$ and (1.13), we find that

$$\int H_{Q_1} f^{(N)}\, d\Omega_1 = \frac{Q_1 f^{(1)}}{\pi\rho\sigma^{1/2}} - \sum_{j\neq 1} \int \frac{Q_j f^{(2,j)}}{\pi\rho r_{1j}}\, d\Omega_{2j}. \qquad (2.4)$$

Here $d\Omega_{2j} = d\boldsymbol{x}_j\, d\boldsymbol{p}_j\, d\sigma_j\, dQ_j$, and the two-particle distribution function $f^{(2,j)}$ is given by the equality

$$f^{(2,j)}(t, \boldsymbol{x}_1, \boldsymbol{p}_1, \sigma_1, Q_1, \boldsymbol{x}_j, \boldsymbol{p}_j, \sigma_j, Q_j) = \int f^{(N)}\, d\Omega_1^{(j)},$$

where $d\Omega_1^{(j)}$ is obtained from $d\Omega_1$ by dropping terms that enter into $d\Omega_{2j}$. Hence, to describe the evolution of the one-particle distribution function, it is necessary to write the equation for the two-particle distribution function, etc. Therefore, additional assumptions are usually invoked to close the equations. As in [4], we use the hypothesis of "molecular chaos," according to which $f^{(2,j)} = f^{(1)}(t, \boldsymbol{x}_1, \boldsymbol{p}_1, \sigma_1, Q_1) f^{(1)}(t, \boldsymbol{x}_j, \boldsymbol{p}_j, \sigma_j, Q_j)$. This allows us to express the second term on the right side of (2.4) as

$$-\sum_{j\neq 1} \int \frac{Q_j f^{(2,j)}}{\pi\rho r_{1j}}\, d\Omega_{2j} = -4(N-1)\,\frac{f^{(1)}}{\rho} \int \frac{K_1}{4\pi|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\, d\boldsymbol{x}_2,$$

where $K_1(t, \boldsymbol{x}_2) = \displaystyle\int Q f^{(1)}(t, \boldsymbol{x}_2, \boldsymbol{p}, \sigma, Q)\, d\boldsymbol{p}\, d\sigma\, dQ$. We note that $-(4\pi|\boldsymbol{x}_1 - \boldsymbol{x}_2|)^{-1}$ is Green's function for the Laplace equation in $\mathbb{R}^3$. Therefore,

$$\int H_{Q_1} f^{(N)}\, d\Omega_1 = \Big(\frac{Q_1}{\pi\rho\sigma_1^{1/2}} + \frac{4(N-1)\psi}{\rho}\Big) f^{(1)}, \quad \Delta\psi = K_1.$$

Since

$$H_{\boldsymbol{x}_1} = -\frac{Q_1}{\pi\rho}\sum_{j\neq 1} Q_j \nabla_{\boldsymbol{x}_1}(r_{1j})^{-1},$$

similarly we obtain

$$\int H_{\boldsymbol{x}_1} f^{(N)}\, d\Omega_1 = 4Q_1(N-1) f^{(1)}\nabla\psi/\rho.$$

We note that H_{σ_1} depends only on the coordinates and momenta of the first particle and, therefore,

$$\int H_{\sigma_1} f^{(N)}\, d\Omega_1 = H_{\sigma_1} f^{(1)}.$$

Substituting the expressions obtained in (2.3), we write the kinetic equation for the one-particle distribution function:

$$f_t^{(1)} + \operatorname{div}_{\boldsymbol{x}}\Big(\frac{2\boldsymbol{p}f^{(1)}}{\tau\rho}\Big) - 4(N-1)\operatorname{div}_{\boldsymbol{p}}\Big(Q\,\frac{f^{(1)}\nabla\psi}{\rho}\Big)$$

$$+ ((Q(\rho\pi\sigma^{1/2})^{-1} + 4(N-1)\rho^{-1}\psi) f^{(1)})_\sigma - (H_{\sigma_1} f^{(1)})_Q = 0.$$

Letting $f(t,\boldsymbol{x},\boldsymbol{p},\sigma,Q) = N f^{(1)}(t,\boldsymbol{x},\boldsymbol{p},\sigma,Q)$ and passing to the limit $N\to\infty$, we obtain the kinetic equations

$$f_t + \operatorname{div}_{\boldsymbol{x}}(H_{\boldsymbol{p}} f) - \operatorname{div}_{\boldsymbol{p}}(H_{\boldsymbol{x}} f) + (H_Q f)_\sigma - (H_\sigma f)_Q = 0 \qquad (2.5)$$

with the Hamiltonian

$$H = \frac{|\boldsymbol{p}|^2}{\rho\tau} + \frac{Q^2}{2\rho\pi\sigma^{1/2}} + \frac{4\psi Q}{\rho} + \varepsilon(\tau). \qquad (2.6)$$

Here $\tau = (4/3)\pi\sigma^{3/2}$ and the function ψ, which defines the self-consistent field, satisfies the Poisson equation

$$\Delta\psi = K, \qquad K = \int Qf\, d\boldsymbol{p}\, d\sigma\, dQ. \qquad (2.7)$$

Equations (2.5)–(2.7) describe the evolution of the one-particle distribution function of compressible bubbles moving in an inviscid incompressible fluid. This system can be treated as a model system that takes into account the collective interaction of bubbles in a first approximation. To improve the accuracy of approximation, it is necessary to consider the next terms in the asymptotic expansion of the kinetic energy of the fluid.

Because Eqs. (1.15) admitted integrals, Eqs. (2.5)–(2.7) obey the hydrodynamic conservation laws

$$\Big(\int f\, d\boldsymbol{p}\, dQ\, d\sigma\Big)_t + \operatorname{div}_{\boldsymbol{x}}\Big(\int H_{\boldsymbol{p}} f\, d\boldsymbol{p}\, dQ\, d\sigma\Big) = 0,$$

$$\Big(\int \boldsymbol{p} f\, d\boldsymbol{p}\, dQ\, d\sigma\Big)_t + \operatorname{div}_{\boldsymbol{x}}\Big(\int (\boldsymbol{p}\otimes H_{\boldsymbol{p}}) f\, d\boldsymbol{p}\, dQ\, d\sigma + 4\rho^{-1}\nabla\psi\otimes\nabla\psi - 2\rho^{-1}|\nabla\psi|^2 I\Big) = 0, \qquad (2.8)$$

$$\Big(\int Hf\, d\boldsymbol{p}\, dQ\, d\sigma + 2\rho^{-1}|\nabla\psi|^2\Big)_t + \operatorname{div}_{\boldsymbol{x}}\Big(\int HH_{\boldsymbol{p}} f\, d\boldsymbol{p}\, dQ\, d\sigma - 4\rho^{-1}\psi_t\nabla\psi\Big) = 0.$$

Here the divergence of the tensor T is defined as follows: $\operatorname{div} T\cdot\boldsymbol{a} = \operatorname{div}(T^*\langle\boldsymbol{a}\rangle)$, where $\boldsymbol{a}$ is an arbitrary vector and T^* is the adjoint mapping [13]. A consequence of Eqs. (2.8) is the law of conservation of angular momentum:

$$\Big(\int (\boldsymbol{x}\times\boldsymbol{p}) f\, d\boldsymbol{p}\, dQ\, d\sigma\Big)_t + \operatorname{div}_{\boldsymbol{x}}\Big(\int ((\boldsymbol{x}\times\boldsymbol{p})\otimes H_{\boldsymbol{p}}) f\, d\boldsymbol{p}\, dQ\, d\sigma + 4\rho^{-1}(\boldsymbol{x}\times\nabla\psi)\otimes\nabla\psi\Big) - 2\rho^{-1}\operatorname{rot}(\boldsymbol{x}|\nabla\psi|^2) = 0.$$

The conservation laws (2.8) can be employed to derive the equations in a hydrodynamic approximation. As an additional law, we can take the following inhomogeneous conservation law:

$$\Big(\int Qf\, d\boldsymbol{p}\, dQ\, d\sigma\Big)_t + \operatorname{div}_{\boldsymbol{x}}\Big(\int QH_{\boldsymbol{p}} f\, d\boldsymbol{p}\, dQ\, d\sigma\Big) + \int H_\sigma f\, d\boldsymbol{p}\, dQ\, d\sigma = 0.$$

3. Steady Solutions. It is easy to see that if f does not depend on t, the equality $f = \Phi(H)$ specifies an exact solution of Eq. (2.5). And for the function ψ, we obtain the equation

$$\Delta\psi = L(\psi), \qquad L(\psi) = \int Qf(H)\, d\boldsymbol{p}\, d\sigma\, dQ.$$

In the one-dimensional case where $\psi_{xx} = L(\psi)$, this equation can be integrated once: $\psi_x^2/2 = \chi(\psi)$, where $\chi(\psi) = \int L(\psi)\, d\psi$, and reduced to quadratures. By giving concrete expressions for the function $f(H)$, it is possible to obtain various solutions including periodic waves.

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